

# Periphony: With-Height Sound Reproduction \*

MICHAEL A. GERZON

*Mathematical Institute, University of Oxford,  
Oxford, England*

Periphony (sound reproduction in both vertical and horizontal directions around a listener) may be recorded among others, via practical two-, four-, and nine-channel systems. Matrix parameters and microphone techniques are described for 19 different systems, and a design procedure for other periphonic systems is given. Amplitude and energy directional resolution are discussed, as is compatibility with current horizontal-only systems.

**INTRODUCTION:** In a recent paper [1], Cooper and Shiga observed that systems of recording sounds that treat all horizontal directions equally form a hierarchy of systems having two, three, four, five, . . . channels, such that every system can be embedded into a system higher in the hierarchy by adding the requisite number of channels. Their methods were those of harmonic analysis, and this paper aims to present the corresponding results, discovered independently, for those systems of sound recording that treat all directions, both horizontal and vertical, equally. Such systems of recording both the horizontal and vertical directional effect are termed "periphonic."

The first periphonic system to be announced was the four-channel tetrahedral system of G. Cooper [2], and further work on such four-channel systems has been done by Bruck [3] and the author [4], [5]. Two-channel periphony has been discussed by Scheiber [6] and the author [7], [8]. Readers are referred to [5] for a survey of the

proposed four-channel periphonic systems.

The reproduction of sound with height can in principle be achieved via any arrangement of loudspeakers that forms a solid, enclosing the listener. The arrangement of Fig. 1 is the simplest arrangement that takes into account various psychoacoustic requirements [8], and recordings suitable for this arrangement will also give satisfactory results via the more usual horizontal square-speaker layout. Although irregular speaker layouts are likely to be used for domestic with-height sound reproduction when it arrives, it is convenient here to consider only fairly regular arrangements of speakers, e.g., at the vertices, face centers or edge centers of an Archimedean solid. Such regular speaker layouts mean that to reproduce sounds from their correct direction in space, each speaker need only be fed with the output of a directional microphone pointing in that direction.

To reproduce all directions perfectly, one would need an infinite number of minute speakers on a sphere around the listener, and an infinite number of "delta-function" (i.e., infinitely directional) microphones pointing in all directions to record the sounds. In practice, the restricted number of channels will reduce the obtainable direc-

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tional resolution. In periphony, we aim to record the values of the sound pressure on the surface of a sphere (representing the directions around the microphones). In television transmission, the problem of transmitting parameters (in that case brightness) over a surface (in that case a rectangle) has been solved by creating a large number of channels (one for each picture element) by time-multiplexing. In periphony, the small number of speakers means that only a few channels are required to convey a low directional resolution. Thus no "scanning" of the sphere is required, and the topological properties of the sphere (notably the "hairy ball theorem") make such scanning in any case undesirable. In order to discuss directional resolution on the sphere, one needs some analog to the theory of Fourier transforms on a line, and this will be provided by the theory of spherical harmonics and spin harmonics in this paper.

### THE COOPER-SHIGA HIERARCHY

Any system of recording horizontal sounds that treats all horizontal directions equally can be shown to be of the following form [1]. Let the number of channels be  $n$  and let  $m_1, m_2, \dots, m_n$  be integers. Then if  $\theta$  is the horizontal angle of arrival of a sound, it is recorded in the  $i$ th channel with gain  $\exp[j(m_i\theta - \alpha(\theta))]$  for all  $i = 1, 2, \dots, n$ , where a gain of  $j = \sqrt{-1}$  means that a signal is subjected to a  $90^\circ$  phase shift, and where  $\alpha(\theta)$  is a phase shift dependent only on the direction of arrival of a sound and not on the channel. If we regard two methods of recording horizontal sounds as being equivalent as long as one method can be turned into the other by passing the  $n$ -channel signal through an  $n \times n$  matrix, then the above describes all directionally unbiased methods of recording sounds horizontally.

A practical recording system will not allow channels to be wasted by duplicating information, and will not omit information conveying coarse directional resolution while transmitting fine-resolution information. By these criteria, practical horizontal systems have  $m_i = i$  for all  $i$ , and usually have  $\alpha(\theta) = \frac{1}{2}(n+1)\theta$  if  $n$  is odd, or  $\alpha(\theta) = -\frac{1}{2}n\theta$  if  $n$  is even. It will be seen that these systems form a hierarchy, with the  $n$ -channel system being convertible to the  $(n+1)$ -channel system merely by adding an additional channel with gain  $\exp[-\frac{1}{2}nj\theta]$  if  $n$  is even or  $\exp[\frac{1}{2}(n+1)j\theta]$  if  $n$  is odd. By sampling theorem arguments [1], it can be shown that the obtainable directional resolution is proportional to the number  $n$  of channels.

### SPHERICAL HARMONIC HIERARCHY

The above results of Cooper and Shiga for horizontal surround-sound systems have analogs for periphonic systems. In order to simplify the required theory of spherical harmonics, we use a "direction cosine" representation for directions in space.

Consider a sphere  $x^2 + y^2 + z^2 = 1$  of unit radius, where the  $x$  axis points forward, the  $y$  axis points to the left, and the  $z$  axis upward. It is evident that any direction in space may be described by its direction cosines, which are defined to be the coordinates  $(x, y, z)$  of the point in that direction from the center of the unit sphere on the unit sphere.

The gain with which a sound from the direction

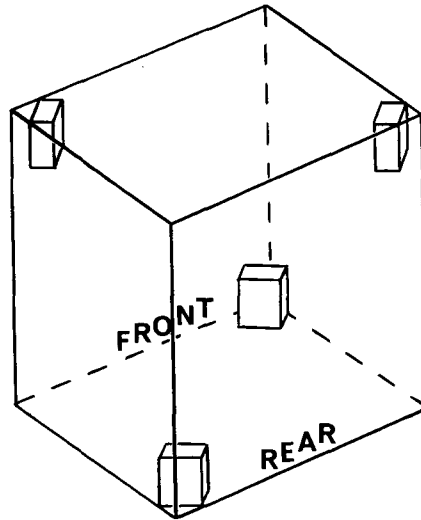


Fig. 1. Tetrahedral loudspeaker layout (embedded in a cube).

$(x, y, z)$  is applied to a channel of the periphonic recording will be a complex function of the direction, i.e., a function on the unit sphere. It is convenient to think of this function as the directional characteristic (i.e., the amplitude gain in various directions) of a directional microphone, although the recording may in fact be made by "pan-potting" sounds into the channels to stimulate pickup by directional microphones.

A function  $f(x, y, z)$  on the sphere  $x^2 + y^2 + z^2 = 1$  is said to have an "energy"

$$\frac{1}{4\pi} \int |f(x, y, z)|^2 ds$$

where the integration is over the surface of the sphere ( $ds$  being an element of that surface), and where the factor  $1/4\pi$  is conveniently adopted to allow for the sphere having surface area  $4\pi$ . A microphone with directional characteristic  $f(x, y, z)$  will pick up directionally homogeneous sounds (e.g., reverberation) with this energy, and its directivity factor  $\gamma$  is defined to be the ratio of the maximum value of  $|f(x, y, z)|^2$  to the "energy" of  $f(x, y, z)$ . As it measures the microphone's susceptibility to directional and directionless sound,  $\gamma$  is a measure of the spatial resolution, with a higher value for more directional microphones. For example, a cardioid microphone has  $\gamma = 3$ , and an omnidirectional one has  $\gamma = 1$ .

Two functions  $f(x, y, z)$  and  $g(x, y, z)$  on the unit sphere are said to be orthogonal if  $\int f(x, y, z) * g(x, y, z) ds = 0$ , where  $*$  means taking complex conjugates. Orthogonality is equivalent to the energy of the sum of  $f$  and  $g$  being equal to the sum of the separate energies of  $f$  and of  $g$ . A similar additive property for energy ensures that the harmonics of a sound waveform are orthogonal to one another, and it is by analogy with these that one defines spherical harmonics on the sphere.

Spherical harmonics are functions  $f(x, y, z)$  on the unit sphere with the following properties:

(0) A zero-order spherical harmonic is a constant function on the sphere, i.e.,  $f(x, y, z) = a$ .

(1) A first-order spherical harmonic is a polynomial function of degree 1 in  $x, y$ , and  $z$  that is orthogonal to all zero-degree polynomials, i.e.,  $f(x, y, z) = ax + \beta y + \gamma z$ .

(n) In general, an  $n$ th-order spherical harmonic (con-

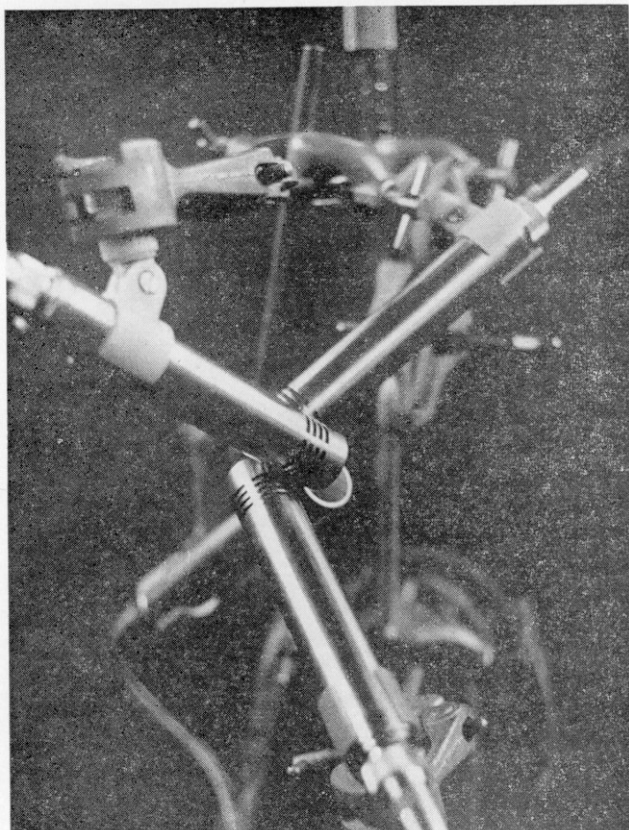


Fig. 2. Tetrasonic microphone arrangement.

veniently termed an  $n$ th harmonic) is a polynomial of degree  $n$  in  $x, y$ , and  $z$  that is orthogonal to all lower degree polynomials.

Examples of second-order spherical harmonics are:  $x^2 - 1/3$ ,  $xy$ ,  $y^2 - z^2$ .

Microphones with a zero-order directional characteristic are omnidirectional, those with a first-harmonic directional characteristic are figure of eight, and clover-leaf microphones [9] have a second-harmonic directional characteristic.

The following results hold for spherical harmonics.

1) The effect of rotating an  $n$ th harmonic is to obtain another  $n$ th harmonic.

2) A linear combination of  $n$ th harmonics is an  $n$ th harmonic. All  $n$ th harmonics may be expressed as a linear combination of rotated versions of any particular non-zero  $n$ th harmonic.

3) Any function on the sphere is expressible, in a unique manner, as the sum of a zeroth harmonic, a first harmonic, a second harmonic, . . . , an  $n$ th harmonic, . . .

4) The number of linearly independent  $n$ th harmonics is  $2n + 1$ , i.e., any  $n$ th harmonic can be expressed as a linear combination of  $2n + 1$  specified  $n$ th harmonics.

The properties 1) to 3) are obeyed by collections of functions on the sphere only if they are spherical harmonics. These properties also make them ideal for use as matrixing and microphone characteristics for periphery. If a recording is made in which one microphone picks up sound with a zeroth-order directional characteristic, three microphones pick up sounds with independent first-harmonic directional characteristics, and so on up to  $2n + 1$  microphones picking up sounds with independent  $n$ th-harmonic directional characteristics, then

the resultant  $1 + 3 + \dots + (2n + 1) = (n + 1)^2$  channels pick up sound in a manner that is essentially independent of direction, by properties 1) and 2). Furthermore, in principle, by making  $n$  large enough, the sound field around the microphones may be captured to any desired degree of accuracy because of property 3).

The system with  $n = 1$ , using  $1 + 3 = 4$  channels, has been termed tetrasonic, and has been the subject of considerable experiment [2], [3], [5]. The microphone arrangement used for this system consists of four coincident microphones, each having a pickup involving both zeroth and first harmonics (e.g., a cardioid or hypercardioid characteristic) pointing in different and non-coplanar directions. For example, tetrasonic microphone arrangement of Fig. 2 has four cardioid or hypercardioid microphones pointing along the four axes of a regular tetrahedron [4], [5].

The  $1 + 3 + 5 = 9$  channel system of periphery, called enneaphony, uses at least nine coincident microphones involving omnidirectional, figure of eight, and clover-leaf pickup. Although full details cannot be given here, such a directional pickup can be obtained using twelve small cardioid or hypercardioid capsules mounted to form the faces of a regular dodecahedron (see Fig. 3) having a small diameter (not more than 5 cm), and the second-harmonic aspects of the directional pickup can be derived from these by techniques similar to the Blumlein difference technique [9]. It seems that the third-harmonic pickup required for 16-channel periphery is more difficult to derive using currently available microphones.

Clearly, each  $n^2$ -channel system of periphery can be upgraded to  $(n + 1)^2$  channels simply by adding  $2n + 1$  channels conveying the  $n$ th-harmonic sound pickup. Thus the 1, 4, 9, 16, 25, . . . channel periphonic systems form a hierarchy of systems, each of which can be embedded in all systems higher in the hierarchy by adding channels.

The actual choice of directional pickup assigned to each of the  $2n + 1$   $n$ th-harmonic channels is arbitrary as long as a matrixing can be devised to recover any

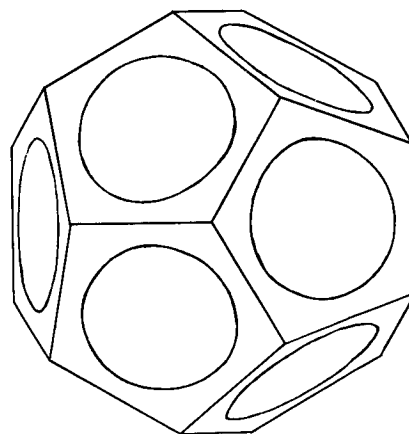


Fig. 3. Arrangement of twelve cardioid or hypercardioid microphone capsules suitable for recording nine-channel periphery. The sound should be permitted access to the rear of the capsules to ensure their pressure-gradient operation.

Table I. Gains of channels of a periphonic recording system based on spherical harmonics, expressed in terms of direction  $(x, y, z)$  of the incident sound

Spin	Channel Gains
0	1
1	$3^{1/2}x, 3^{1/2}y, 3^{1/2}z$
2	$\frac{1}{2} 15^{1/2}(x^2-y^2), 15^{1/2}xy, \frac{1}{2} 5^{1/2}(3z^2-1), 15^{1/2}xz, 15^{1/2}yz$
3	$(35/8)^{1/2}x(x^2-3y^2), (35/8)^{1/2}y(y^2-3x^2), (21/8)^{1/2}x(1-5z^2), (21/8)^{1/2}y(1-5z^2), 105^{1/2}xyz, \frac{1}{2} 105^{1/2}(x^2-y^2)z, \frac{1}{2} 7^{1/2}(5z^3-3z)$

The order of the spherical harmonics in each line is given in the column headed Spin. The coefficients shown make each channel have a pickup energy of 1, and each channel is orthogonal to all others. A practical periphonic system will include all channels having a spin less than some chosen value. If only the first two channels given for each spin are recorded, then a horizontal-only system in the Cooper-Shiga hierarchy is obtained. The periphonic systems in this table are those belonging to the "weight O" hierarchy (Fig. 5).

$n$ th harmonic pickup from these. Table I gives suitable pickup characteristics for each channel. The first two pickup characteristics given for each order of harmonic greater than zero do not involve  $z$ , and thus by recording the zeroth-harmonic channel and the first two channels in Table I of each of the first to  $n$ th harmonics, a horizontal recording system is obtained using  $2n + 1$  channels. Thus it will be seen that the  $(2n + 1)$ -channel system in the Cooper-Shiga hierarchy of horizontal-only systems can be embedded into the  $(n + 1)^2$ -channel periphonic system belonging to the spherical harmonic hierarchy by adding the channels in Table I involving  $z$  in their directional pickup.

From property 3) of spherical harmonics, the more channels a periphonic system has, the more resolution it has in picking up directional information. This is seen clearly from Fig. 4, which shows the directional characteristics obtainable by matrixing information from the 1, 4, 9, and 16-channel periphonic systems that have the highest possible directivity factors. There is a very general theorem which directly relates the number of channels to the obtainable directional resolution, known as the directivity factor theorem.

### Directivity Factor Theorem

Let a system of  $N$ -channel sound recording provide for pickup of sound-energy information in a directionally unbiased fashion and suppose that there be no duplication of information among the channels (i.e., that no channel be a linear combination of the others). Then the directivity factor of the directional characteristic of energy pickup of any linear combination of the  $N$  channels is not greater than  $N$ . It is possible to find matrixings of the  $N$  channels with a directivity factor equal to  $N$ , and in that case the energy pickup characteristic will be symmetric about an axis that may point in any chosen direction.

The theorem applies either to periphonic or horizontal-only systems, as long as the definition of "directivity factor" in the latter case involves integration over a horizontal circle rather than a sphere.

### PERIPHONIC SYSTEMS OF NONZERO WEIGHT

As no other class of functions on the sphere obeys all the properties 1)-3) for spherical harmonics, it at first appears that all directionally unbiased periphonic systems have been classified. However, other periphonic systems exist, notably the two-channel system devised in-

dependently by Scheiber [6] and the present author (see Appendix I, [7], [10]).

These systems arise by considering the pickup of sound energy rather than sound amplitude. If  $A(t)$  is an audio signal, where  $t$  is time, then its energy is defined to be the quantity  $\int A(t)^2 dt$ , where the integration is over a "long" period of time. In an  $N$ -channel recording, the energy of any signal obtained by taking (real) linear combinations of channels can be shown to be expressible as a linear combination of the  $\frac{1}{2}N(N+1)$  energy parameters of the form  $\int A_i(t)A_k(t)dt$ , where  $A_i$  and  $A_k$  are

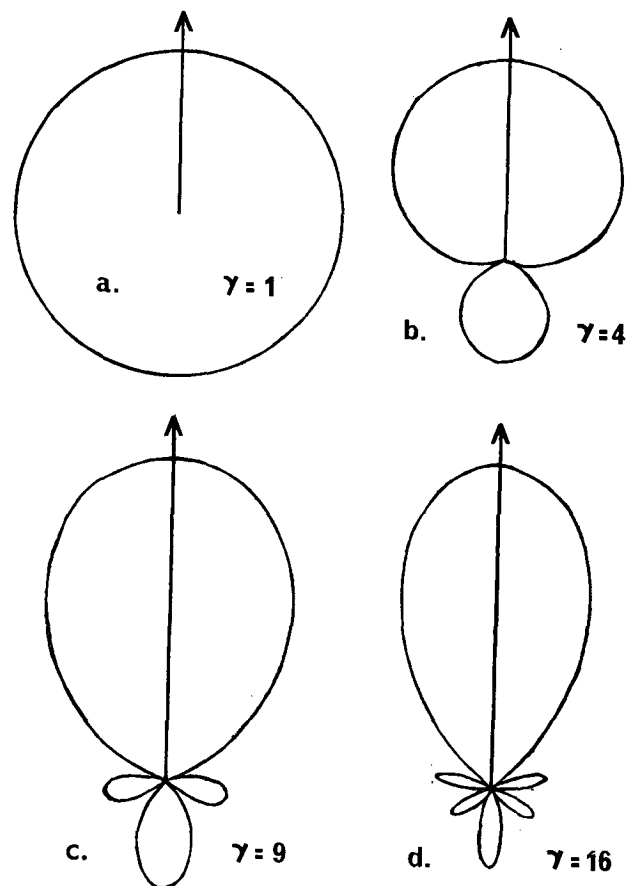


Fig. 4. Amplitude pickup characteristics with maximum directivity factor  $\gamma$  with multichannel periphonic systems based on spherical harmonics. a. One channel. b. Four channels. c. Nine channels. d. Sixteen channels.

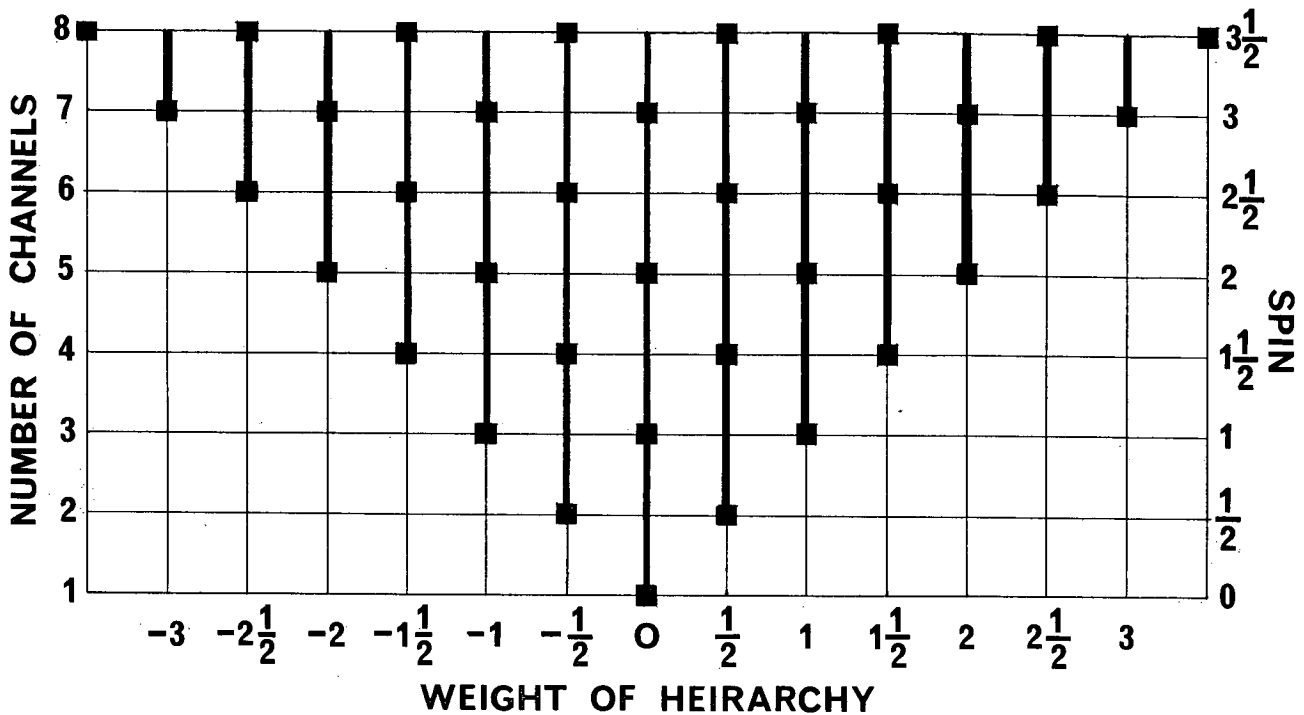


Fig. 5. Hierarchical structure of periphonic systems. Each boldface vertical line represents a distinct hierarchy of systems, and each square an "irreducible" system with the indicated number of channels, spin, and weight. Practical systems consist of simultaneously recording all irreducible systems of a given weight that have a spin larger than or equal to some chosen value of spin. For example, a practical weight 1 system might incorporate spins 1 and 2 and will then have  $3 + 5 = 8$  channels.

the audio signals on the  $i$ th and  $k$ th channels. When (complex) linear combinations involving  $90^\circ$  phase shifts are used, the energy can be shown to be a linear combination of the  $N^2$  energy parameters of the form  $\int A_i(t) A_k(t) dt$  or (when  $i < k$ )  $\int A_i(t) J A_k(t) dt$ , where  $J$  is the  $90^\circ$  phase shift operator. Note that if a signal passes through a recording system with a complex amplitude gain  $u + jv$ , then its energy gain is  $u^2 + v^2 = |u + jv|^2$ . It will be convenient to call  $|u + jv|^2$  the "absquare" of  $u + jv$ .

If a periphonic recording is made so that sound energy is picked up in a directionally unbiased manner, then sound amplitude information need not be picked up in an unbiased manner, as the phase of the sound pickup will depend on the direction of the sound. Energy will be picked up in a directionally unbiased manner if for any directional pickup  $f(x, y, z)$  obtainable by taking complex linear combinations of the channels of the periphonic recording, and for any rotation  $R$ , there is a directional pickup  $g(x, y, z)$  similarly obtainable whose absquare equals the absquare of  $f(x, y, z)$  rotated by  $R$ . The problem of determining all such periphonic systems has been formulated in group-theoretic terms (see Appendix II). The solution to this problem may be stated in terms of a generalization of spherical harmonics, called "spin harmonics" due to their connection with the quantum theory of particles with spin.

Spin harmonics are functions on the unit sphere, and may be of many different "types," one special type being ordinary spherical harmonics. All spin harmonics of a given type belong to one and only one set of functions in a sequence of sets of spin harmonics of that type; functions in any set in the sequence are said to have a different spin from those in other sets in the sequence. (Spherical harmonics have the same spin only if they have the same order). For spin harmonics of a given type, the following properties hold.

- 1) Any two spin harmonics of differing spin are orthogonal.
- 2) Any linear combination of spin harmonics having a given spin is also a spin harmonic having that spin.
- 3) When absquared and rotated, any spin harmonic  $f(x, y, z)$  on the sphere becomes the absquare of another spin harmonic  $g(x, y, z)$  having the same spin. If  $f(x, y, z)$  is nonzero, then any spin harmonic having its spin is expressible as a linear combination of such  $g(x, y, z)$ 's.
- 4) Any function on the sphere is expressible uniquely as the sum of a series of spin harmonics, each having a spin different from the others.

The spherical harmonics are one special type of spin harmonic. In general, spin harmonics differ from spherical harmonics in that only the absolute value of functions is rotatable. If we choose a sequence of sets of func-

Table II. Channel gains of weight  $\frac{1}{2}$  periphonic recording systems as a function of direction  $(x, y, z)$

Spin	Channel Gains
$\frac{1}{2}$	$(1+z)^{1/2}, (1+z)^{-1/2}(x-jy)$
$1\frac{1}{2}$	$(3/2)^{1/2}(1+z)^{1/2}(x+jy), 2^{-1/2}(1+z)^{1/2}(3z-1), 2^{-1/2}(1+z)^{-1/2}(3z+1)(x-jy), (3/2)^{1/2}(1+z)^{-1/2}(x-jy)^2$
$2\frac{1}{2}$	$(15/8)^{1/2}(1+z)^{1/2}(x+jy)^2, (3/8)^{1/2}(1+z)^{1/2}(x+jy)(5z-1), (3/4)^{1/2}(1+z)^{1/2}(5z^2-2z-1), (3/4)^{1/2}(1+z)^{-1/2}(x-jy)(5z^2+2z-1), (3/8)^{1/2}(1+z)^{-1/2}(x-jy)^2(5z+1), (15/8)^{1/2}(1+z)^{-1/2}(x-jy)^3$

Table III. Channel gains of weight 1 periphonic recording systems as a function of direction  $(x, y, z)$ 

Spin	Channel Gains
1	$\frac{1}{2} 3^{1/2} (1+z)^{-1} (x-jy)^2$ , $(3/2)^{1/2} (x-jy)$ , $\frac{1}{2} 3^{1/2} (1+z)$
2	$\frac{1}{2} 5^{1/2} (1+z)^{-1} (x-jy)^3$ , $\frac{1}{2} 5^{1/2} (1+z)^{-1} (2z+1)(x-jy)^2$ , $(15/2)^{1/2} z(x-jy)$ , $\frac{1}{2} 5^{1/2} (1+z)(2z-1)$ , $\frac{1}{2} 5^{1/2} (1+z)(x+jy)$
3	$\frac{1}{8} 105^{1/2} (1+z)^{-1} (x-jy)^4$ , $\frac{1}{4} (35/2)^{1/2} (1+z)^{-1} (3z+1)(x-jy)^3$ , $\frac{1}{8} 7^{1/2} (1+z)^{-1} (15z^2+10z-1)(x-jy)^2$ , $\frac{1}{4} 21^{1/2} (1-5z^2)(x-jy)$ , $\frac{1}{8} 7^{1/2} (1+z)(15z^2-10z-1)$ , $\frac{1}{4} (35/2)^{1/2} (1+z)(3z-1)(x+jy)$ , $\frac{1}{8} 105^{1/2} (1+z)(x+jy)^2$

tions on the sphere obeying 1)–4) above, we also define a hierarchy of periphonic systems. A typical periphonic system in this hierarchy has the following form. Choose a set of spins. For each spin in the chosen set, record a number of channels, each of which picks up sound from each direction with a gain that is a spin harmonic having that spin, in a manner such that there is no duplication of information among the channels and so that every spin harmonic pickup of that spin can be obtained by giving the signals of the channel complex gains (i.e., involving  $90^\circ$  phase shifts) and adding them. The periphonic recording will consist of all the channels conveying sound picked up with spin harmonics having a spin in the chosen set of spins.

Just as spherical harmonic periphonic systems were built out of “irreducible” systems consisting of all the channels required to convey all pickup characteristics of one specific order, so the more general periphonic systems are built out of “irreducible” systems consisting of all the channels required to convey all spin harmonic pickups having one specific spin. Thus periphonic systems using spin harmonics of a given type form a hierarchy of systems, in which the spatial resolution of each system can be improved by adding channels conveying pickups having other spins.

In order to see what spin harmonics look like as functions on the sphere, it is necessary to state a number of properties that apply to any type of spin harmonic, but which cannot be proved here.

1) The number of channels required to convey all the spin harmonic pickups of any one spin without duplication of information differs from that required to convey those of any other spin by a nonzero even number.

2) Let the number of channels required to convey spin harmonic pickups of a particular spin be  $2s + 1$ , then the number  $s$  is called the “spin” of these spin harmonics. Let  $S$  be the smallest spin obtainable with a given type of spin harmonic. Then spin harmonics exist having all the spins possible according to 1), i.e.,  $S, S + 1, S + 2, \dots, S + n, \dots$

Of the various possible types of spin harmonics that can be found obeying the properties 1)–4) given earlier, some types may be obtained from other types of spin harmonics by multiplying the latter functions on the sphere by a phase factor (i.e., function of modulus 1)  $\exp[ja(x, y, z)]$ , where  $a$  is a real function. Two types of spin harmonics thus related will give periphonic recordings that differ only in the absolute phase with which

sounds from different directions are recorded. As the ear is tolerant of such phase shifts, there is no need to distinguish between such systems.

Every type of spin harmonic may be assigned a number  $w$  (called its weight) which is a negative, zero, or positive integer multiple of  $\frac{1}{2}$ , such that the smallest spin occurring for that type of spin harmonic is  $|w|$ , and such that types of spin harmonic having the same weight differ only by a phase factor as described above.

What has been shown above is that every hierarchy of periphonic systems is described by a number  $w$  that is a multiple of  $\frac{1}{2}$ , and that every periphonic system is composed out of “irreducible” systems having a spin of the form  $|w| + n$  (where  $n$  is an integer  $\geq 0$ ). For example, the spherical harmonic systems have a weight 0 and are composed out of systems having nonnegative integer spins; in this case the “spin” is equal to the order of the spherical harmonic involved. Fig. 5 shows the above hierarchical relationships between periphonic systems, where each dot represents an irreducible spin system.

Practical periphonic systems will usually consist of the  $(2|w|+1) + (2|w|+3) + \dots + (2|w|+2n+1) = (n+1)(2|w|+n+1)$  channels required to convey the spin harmonics having the  $n$  lowest spins in the weight  $w$  hierarchy. Tables II–IV give the actual pickup characteristics required to record periphonic sounds using, respectively, spin harmonics of weight  $\frac{1}{2}$ , 1 and  $1\frac{1}{2}$ . (Table I, of course, gives these for weight 0). In all these tables, the pickup of all channels has energy 1 and is orthogonal to that of all other channels in each hierarchy. Any practical periphonic system will incorporate all channels in its hierarchy having a spin less than or equal to a chosen value. Note that  $90^\circ$  phase shifts are an essential feature of the nonzero-weight systems. The channel gains of systems with weight  $-w$  may be obtained from those of weight  $w$  by changing  $j$  to  $-j$ , i.e., by replacing all  $90^\circ$  phase shifts by  $-90^\circ$  phase shifts.

It is convenient to denote the periphonic system of weight  $w$  and using spins  $s_1, s_2, \dots, s_n$  by the symbolism  $D_w^{s_1} + D_w^{s_2} + \dots + D_w^{s_n}$ . Thus, for example, enneaphony is denoted by  $D_0^0 + D_0^1 + D_0^2$ .

## SPATIAL RESOLUTIONS OF PERIPHONIC SYSTEMS

The directivity factor theorem was formulated so as

Table IV. Channel gains of weight  $1\frac{1}{2}$  periphonic recording systems as a function of direction  $(x, y, z)$ 

Spin	Channel Gains
$1\frac{1}{2}$	$2^{-1/2} (1+z)^{-3/2} (x-jy)^3$ , $(3/2)^{1/2} (1+z)^{-1/2} (x-jy)^2$ , $(3/2)^{1/2} (1+z)^{1/2} (x-jy)$ , $2^{-1/2} (1+z)^{3/2}$
$2\frac{1}{2}$	$\frac{1}{2} 15^{1/2} (1+z)^{-3/2} (x-jy)^4$ , $\frac{1}{4} 3^{1/2} (1+z)^{-3/2} (5z+3)(x-jy)^3$ , $\frac{1}{2} (3/2)^{1/2} (1+z)^{-1/2} (5z+1)(x-jy)^2$ , $\frac{1}{2} (3/2)^{1/2} (1+z)^{1/2} (5z-1)(x-jy)$ , $\frac{1}{4} 3^{1/2} (1+z)^{3/2} (5z-3)$ , $\frac{1}{2} 15^{1/2} (1+z)^{3/2} (x+jy)$

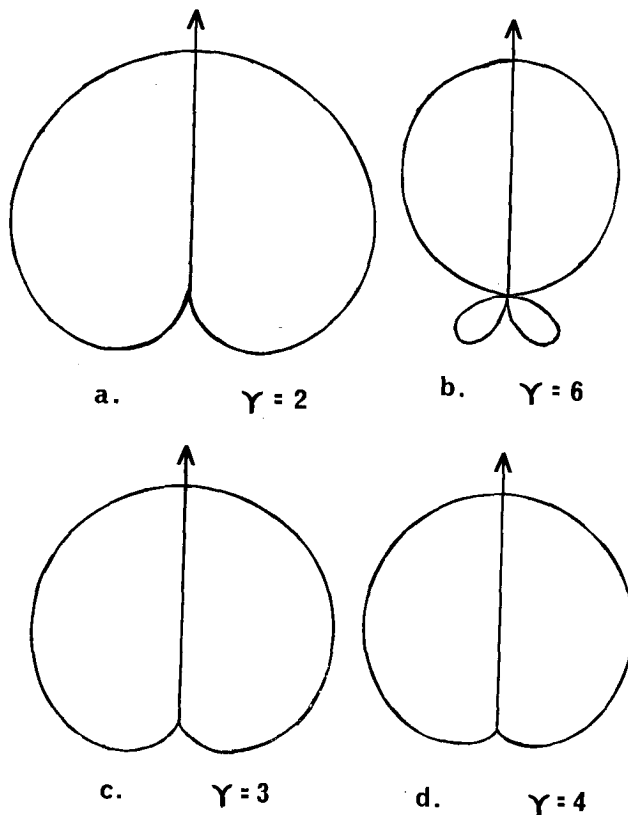


Fig. 6. Amplitude pickup characteristics (ignoring phase) with the maximum directivity factor obtainable with the systems. a.  $D_{1/2}^{1/2}$  with two channels. b.  $D_{1/2}^{1/2} + D_{1/2}^{1/2}$  with six channels. c.  $D_{1/2}^{1/2}$  with three channels. d.  $D_{1/2}^{1/2}$  with four channels.

to include all the periphonic systems discussed above. Thus the two-channel periphonic system  $D_{1/2}^{1/2}$ , having the channel gains in the first line of Table II, can resolve directional information with a directivity factor  $\gamma = 2$ . Similarly,  $D_{1/2}^{1/2} + D_{1/2}^{1/2}$ , whose six channels pick up sound with the gains shown in the first two lines of Table II, is capable of  $\gamma = 6$ , and the three channels of  $D_{1/2}^{1/2}$  can have  $\gamma = 3$ . In Fig. 6 the amplitude directional characteristic (without phase information) is shown of the most directional sound pickups obtainable with the systems  $D_{1/2}^{1/2}$ ,  $D_{1/2}^{1/2} + D_{1/2}^{1/2}$ ,  $D_{1/2}^{1/2}$ ,  $D_{1/2}^{1/2}$ , and  $D_{1/2}^{1/2}$ . It will be noted from Figs. 4b and 6d that both of the systems  $D_{0^0} + D_{0^1}$  and  $D_{1/2}^{1/2}$  have a maximum  $\gamma = 4$ , but that the actual shape of the pickup is different, as the former has a rear-lobe response and the latter does not.

The very general rule for matrixing the channels of a periphonic recording for playback with highest possible directivity factor is as follows. In each of the Tables I to IV the pickup characteristics of the channels have been given with "energy" 1 and mutually orthogonal. In these circumstances, the most directional pickup pointing in the direction  $(p, q, r)$  (with  $p^2 + q^2 + r^2 = 1$ ) is obtained by adding together all the channels of the recording after giving each a suitable gain. The gain given to each channel is obtained from the tables by substituting  $p, q$ , and  $r$  for  $x, y$ , and  $z$  respectively, and substituting  $-j$  for  $j$  in the entry for that channel in Tables I-IV. For example, the two channels of  $D_{1/2}^{1/2}$  (first line of Table II) will be recovered for a speaker lying in the direction  $(p, q, r)$  with a gain  $(1 + r)^{1/2}$  on the first channel and with gain

$(1 + r)^{1/2}(p + jq)$  on the second.

Mathematically, the property of having the maximum directivity during playback coincides with a number of important properties of a sound-recording system, notably that of having minimal interspeaker crosstalk and that of giving the least-squares approximation to perfect directional reproduction within the available number of channels. The highest directivity pickup characteristic can be shown to be the sum of those terms of the spin-harmonic expansion of the delta-function characteristic pointing in its direction that are recorded by the periphonic system being used. The "energy" of each spin harmonic component of this highest-directivity characteristic is proportional to the number of channels used for that spin without duplication of information.

It may well be that the most directional pickup is not the one that gives the best psychoacoustic results. For example, the most directional pickup of the tetraphonic system  $D_{0^0} + D_{0^1}$  (see Fig. 4b) is a hypercardioid with a null  $109.5^\circ$  off axis, and has a rather large rear-lobe; initial experiments [5] suggest that a hypercardioid with a  $135^\circ$  null is subjectively better with the tetrahedral speaker layout of Fig. 1. A pickup of sound energy that is symmetric about an axis but does not have maximum directionality may be obtained as described above, except that the channel gains should be multiplied by a positive constant that is the same for all channels of a given spin, but possibly different for different spins.

The number of speakers required for reproduction depends on the directional resolution of the periphonic system, and should not in any case be less than the number of channels  $N$ . However, while only  $N$  parameters suffice to describe the amplitude in the  $N$  channels, if real matrixing is used to feed the speakers, then  $\frac{1}{2}N(N + 1)$  energy parameters are required to describe their energy, and  $N^2$  energy parameters are required if  $90^\circ$  phase shifts are used as well. In practice, most periphonic systems have some of their energy parameters equal to zero; for example, in tetraphonic recording, the energy of the sound picked up by the omnidirectional characteristic 1 minus the sum of the energies picked up by the three figure-of-eight characteristics  $x, y$ , and  $z$  is zero, as  $1 - x^2 - y^2 - z^2 = 0$ . The number of independent energy parameters actually conveyed by various periphonic systems is given in Table V. For example, tetraphony has nine independent energy parameters, whereas the two-channel system has four. Thus if the spatial distribution of sound energy around the listener implicit in the recording is to be fully captured, many more speakers will be needed than the number of channels. However, many of the energy components corresponding to a high degree of spatial resolution are weak, and so an intermediate number of loudspeakers should prove sufficient.

## PHASE IN PERIPHONY

It will be noted in Table II-IV that if  $(x, y, z) = (0, 0, -1)$ , then the meaningless expression  $0/0$  occurs as a channel gain in all nonzero-weight systems. In the neighborhood of this downward direction, the phase of the sound pickup varies in a discontinuous manner. It may be thought that for each system there is a phase factor  $\exp[ja(x, y, z)]$  which, when multiplied by the chan-

nel gains, removes this discontinuity, but this is not so. The most such a phase factor can do is to move the discontinuity to a different direction, as shown, for example, in Appendix I. The degree of a discontinuity in phase about a direction of discontinuity may be defined as the multiple of  $360^\circ$  by which the phase of those channel gains which are discontinuous changes during a passage near and around the direction of discontinuity. The degree of discontinuity of phase in a periphonic system can be shown to be at least twice the modulus of the weight of the system. Thus the lower weight systems have less discontinuity of phase than those of higher weight.

As shown above, the coefficients of Tables II–IV also describe the playback process, and so the phase discontinuity will cause phase differences to appear between speakers, especially when a large number are used. Only the weight-zero systems are free from this property, and high-weight systems show it particularly strongly.

### OTHER PROPERTIES

Sound may be recorded for weight- $w$  systems using available (and continuous) coincident microphones if their channel gains in Tables II–IV are multiplied by  $(1 + z)^{|w|}$  during recording, so as to remove all divisions by this factor. This will have the effect of recording sounds in the direction  $z = 1$  with increased gain, while diminishing the gains of sounds for which  $z < 0$ . However, as the relative amplitudes and phases of the channels are unchanged, such recordings may still be reproduced via a weight- $w$  playback system with correct directional effect.

This multiplication of channel gains converts them into polynomials in  $x$ ,  $y$ , and  $z$ , so that it is possible to rematrix weight-zero recordings into weight- $w$  recordings, albeit with a directionally biased gain. Thus, for example, multiplying the  $D_{\frac{1}{2}}^{\frac{1}{2}}$  channel gains by  $(1 + z)^{\frac{1}{2}}$  gives channel gains that may be derived by matrixing a  $D_0^0 + D_0^1$  recording. Also, note that if the channel gains of  $D_{\frac{1}{2}}^{\frac{1}{2}}$  (first line, Table II) are multiplied by, say  $(1 + z)^{\frac{1}{2}}(1 - \frac{1}{2}z)$ , then a recording is obtained whose gain is more directionally uniform, and which can be obtained by matrixing a  $D_0^0 + D_0^1 + D_0^2$  recording. In a similar way, it will be seen that a recording suitable for any periphonic system may be obtained by matrixing any recording of lower weight having a sufficient number of spins. Thus the periphonic hierarchies are interrelated, as it is possible to convert low-weight recordings into ones of higher weight.

Any periphonic system may be regarded as a directionally unbiased horizontal-only system by considering only horizontal sounds. Most periphonic systems have duplication of horizontal information among their channels, so that the recordings do not directly belong to the

Cooper–Shiga hierarchy, although they can be made to do so by neglecting those channels which duplicate horizontal information. However, systems of the form  $D_{\pm w}^w$  become horizontal systems in the Cooper–Shiga hierarchy when we put  $z = 0$ ; note that the systems  $D_{\pm w}^w$  and  $D_{\mp w}^w$ , which are nonequivalent periphonic systems, are equivalent as  $(2w + 1)$ -channel horizontal systems in the sense that recordings for one horizontal system may be matrixed for playback via the other.

Thus, at first sight, it seems that horizontal-only systems can have height added only if they belong to the Cooper–Shiga hierarchy. However, it was shown in [4] that the conventional pairwise-mixed four-channel horizontal system (which does not belong to the Cooper–Shiga hierarchy) can have height added. Any horizontal-only system  $H$  can be embedded in a periphonic system having a sufficiently high spatial resolution, as  $H$  microphone characteristics (representing the channel gains as a function of direction) can always be expressed as a sum of spin harmonics. However, height cannot be added in practice if the periphonic system used requires a domestically impractical number of channels (e.g.,  $\geq 9$ ). Also,  $H$  gives rise to incompatibility problems if a three-channel Cooper–Shiga system cannot be matrixed for good results via  $H$ , as a tetraphonic recording gives good horizontal four-channel results [4], but will not then give good results via  $H$ . As an example of such an  $H$ , the CBS “SQ” system fails both criteria, so that it is totally unsuitable for adding height. Most other 4–2–4 systems are compatible with the two-channel periphonic system  $D_{\frac{1}{2}}^{\frac{1}{2}}$ .

### CONCLUSIONS

A considerable variety of periphonic systems exist, and it is possible to rematrix recordings made for one system for many other systems in the general case. It is possible to add height to horizontal systems of the Cooper–Shiga type, and to the conventional horizontal four-channel system [4], [5]. Particular flexibility is obtained if the enneaphonic (nine-channel) periphonic system is used for recording mastertapes, as such recordings are convertible via matrixing into a large variety of systems, including two-channel and four-channel periphony.

A mathematical theory of “spin harmonics” has been presented, analogous to spherical harmonics, to describe the general periphonic systems. This paper has only been able to summarize the properties of these systems, as a detailed account of the practical aspects of any particular system would itself occupy several papers. Among aspects that require detailed discussion and investigation are microphone and domestically acceptable speaker layouts, periphonic psychoacoustics, and the choice of the linear combinations of spin harmonics that are assigned

Table V. Number of channels and independent energy parameters of various periphonic recording systems

System	$D_{\pm \frac{1}{2}}^{\frac{1}{2}}$	$D_{\pm 1}^1$	$D_{\pm 1 \frac{1}{2}}^{1 \frac{1}{2}}$	$D_0^0 + D_0^1$	$D_{\pm \frac{1}{2}}^{\frac{1}{2}} + D_{\pm 1 \frac{1}{2}}^{1 \frac{1}{2}}$	$D_{\pm 1}^1 + D_{\pm 1}^2$	$D_0^0 + D_0^1 + D_0^2$
Number of channels	2	3	4	4	6	8	9
Number of energy parameters	4	9	16	9	16	25	25



to each track of the mastertape. Those who have had the opportunity of hearing periphony at its best can have no doubt that the height effect is important in the perception of sound and the enjoyment of music, and it thus seems worthwhile to ensure that current recording media have the possibility of adding the height effect at some future time.

## APPENDIX I:

### TWO-CHANNEL PERIPHONY

The two-channel periphonic system  $D_{1/2}$  records a signal from the direction  $(x, y, z)$  with the respective gains  $(1+z)^{1/2}$  and  $(1+z)^{1/2}(x-jy)$  on the two channels. For example, the first gain may be that of the sum channel of a stereo recording, and the second that of  $j$  times the difference channel. The signal for a speaker in the direction  $(p, q, r)$  is obtained by adding  $(1+r)^{1/2}$  times the first channel to  $(1+r)^{-1/2}(p+jq)$  times the second.

A recording may be made for this system by multiplying the above gains by  $(1+z)^{1/2}$ . Thus the first channel is picked up by an upward-pointing cardioid  $(1+z)$ , and the second channel by a "turnstile" microphone  $(x-jy)$  (i.e., by a forward-pointing figure of eight added to a  $90^\circ$  phase-shifted sideways figure of eight). Alternatively, this sound pickup can be achieved by any three linearly independent coincident conventional-characteristic microphones having a null downward response, such as a forward-pointing figure of eight and two  $135^\circ$ -null hypercardioids pointing  $45^\circ$  above sideways left and sideways right, suitably matrixed.

The second-channel pickup  $(1+z)^{1/2}(x-jy)$  is discontinuous near  $z = -1$ , but by multiplying the channel gains by a function of  $x, y$ , and  $z$  of modulus 1, it is possible to point the discontinuity in the direction  $(-\sin \theta, 0, -\cos \theta)$  at an angle  $\theta$  behind downward vertical. The two-channel gains for two-channel periphony then become

$$\frac{1}{2} \csc \frac{1}{2} \theta [\sin \theta (1+z) + (1-\cos \theta)(x+jy)] \\ (1+z \cos \theta + x \sin \theta)^{-1/2}$$

and

$$\frac{1}{2} \sec \frac{1}{2} \theta [\sin \theta (1-z) + (1+\cos \theta)(x-jy)] \\ (1+z \cos \theta + x \sin \theta)^{-1/2}.$$

The playback gains may be obtained by substituting  $p, q, r$ , and  $-j$  for  $x, y, z$ , and  $j$ .

## APPENDIX II:

### GROUP-THEORETIC FORMULATION

The results of this paper were obtained using the theory of representations of the rotation group  $S(3)$  [11]. The problem of classifying all directionally unbiased periphonic systems reduces to the following. Let a projective representation of  $S(3)$  (which may always be assumed to be unitary) act on a vector space  $V$ . Then find an onto map from the points of the sphere  $S^2$  to a set  $R$  of rays (i.e., one-dimensional subspaces) in  $V$ , such that the action of  $S(3)$  is preserved. The channel gains of a periphonic system are then the coordinates (with respect to some basis of  $V$ ) of a set of unit vectors, one in each of the rays in  $R$ .

Dr. Keith Hannabus of the Mathematical Institute, Oxford, has derived the following explicit formula for spin harmonics, which has proved to be invaluable in investigating periphonic systems. Let  $e_w$  be a vector of weight  $w$  in the spin- $s$  representation of  $S(3)$ , i.e., a vector that is multiplied by  $e^{jw\theta}$  whenever a rotation about the  $z$  axis by the angle  $\theta$  is applied. Then the ray corresponding to a point  $(x, y, z)$  in the sphere  $S^2$  contains a representative vector  $v$  of the form

$$v(x, y, z) = \sum_{i=0}^{s+w} \sum_{k=0}^{s-w} (-1)^i \begin{pmatrix} s+w \\ i \end{pmatrix} \begin{pmatrix} s-w \\ k \end{pmatrix} \\ (1+z)^{s-i-k} (x-jy)^i (x+jy)^k e_{w-i+k}.$$

For all  $i$ , the coefficient of  $e_i$  is a weight- $w$  spin- $s$  spin harmonic, and all spin harmonics of weight  $w$  and spin  $s$  are linear combinations of these  $s$  coefficients.

Since writing this paper, I have found that "spin harmonics" had previously been invented under the name "spin spherical harmonics" [12] for use in general relativistic cosmology.

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**Note:** Mr. Gerzon's biography appeared in the July/August 1972 issue of the Journal.